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2005 J. Phys. A: Math. Gen. 38 5217

(http://iopscience.iop.org/0305-4470/38/23/009)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 5217–5226 doi:10.1088/0305-4470/38/23/009

Darboux transformation and soliton solutions for the Heisenberg hierarchy

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Received 30 March 2005
Published 25 May 2005
Online at stacks.iop.org/JPhysA/38/5217

Abstract
Starting from a spectral problem, we derive the well-known Heisenberg hierarchy. An explicit and universal Darboux transformation for the whole hierarchy is constructed. The soliton solutions for the Heisenberg hierarchy are obtained by applying the Darboux transformation.

PACS numbers: 05.45.Yv, 02.30.Jr

1. Introduction

From the middle of the 1970s, the continuous Heisenberg spin chain aroused considerable interest [1–5]. Tjon and Wright obtained the explicit formula for the single-soliton solution in the isotropic case [3]. Takhtajan studied the integration of the continuous Heisenberg spin chain equation through the inverse scattering transform method and obtained its Lax representation [4]. Afterwards, Chen and Li gave the higher order Heisenberg spin chain equations, and they proved that these evolution equations are equivalent to the evolution equation of AKNS type [5]. By the use of the spectral problem nonlinearization method, Qiao obtained the finite-dimensional integrable system and the involutive solutions of the higher order Heisenberg spin chain equations [6].

Darboux transformation is a powerful method to get exact solutions of nonlinear partial differential equations. The key for constructing Darboux transformation is to expose a kind of covariant properties that the corresponding spectral problems possess. There have been many tricks to do this for getting explicit solutions to various soliton equations [7–10]. In this paper, we are interested in the Darboux transformation and exact solutions of the Heisenberg hierarchy associated with the following Heisenberg spectral problem:

$$\psi_x = U\psi = \lambda U_1 \psi = \begin{pmatrix} -\lambda\omega & \lambda u \\ \lambda v & \lambda\omega \end{pmatrix} \psi, \qquad \omega^2 + uv = 1. \tag{1.1}$$

The outline of our present paper is as follows. In section 2, we derive the Heisenberg hierarchy associated with the spectral problem (1.1). In section 3, we construct Darboux transformation for the Heisenberg hierarchy. In section 4, we construct soliton solutions for the hierarchy by using its Darboux transformation.

2. The Heisenberg hierarchy

In order to derive the isospectral hierarchy associated with (1.1), we consider the auxiliary problem

$$\psi_t = V^{(n)}\psi, \qquad V^{(n)} = \sum_{k=0}^{n} V_k \lambda^{n-k+1} = \sum_{k=0}^{n} \begin{pmatrix} V_{11}^{(k)} & V_{12}^{(k)} \\ V_{21}^{(k)} & V_{22}^{(k)} \end{pmatrix} \lambda^{n-k+1}. \tag{2.1}$$

The compatibility condition between (1.1) and (2.1) yields the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{2.2}$$

which is equivalent to the following recurrence relations:

$$\begin{aligned} &U_1 V_0 - V_0 U_1 = 0, \\ &V_{k-1,x} = U_1 V_k - V_k U_1, \qquad 1 \leqslant k \leqslant n \\ &U_{1t} = V_{nx}. \end{aligned} \tag{2.3}$$

Further we choose $V_{11}^{(0)} = -\omega$, $V_{12}^{(0)} = u$, $V_{21}^{(0)} = v$, $V_{22}^{(0)} = \omega$, and from (2.3) we have

$$-2\omega V_{12}^{(k)} - u\big(V_{11}^{(k)} - V_{22}^{(k)}\big) = V_{12x}^{(k-1)}, \tag{2.4}$$

$$v\big(V_{11}^{(k)} - V_{22}^{(k)}\big) + 2\omega V_{21}^{(k)} = V_{21x}^{(k-1)}, \tag{2.5}$$

$$\partial_x\big(u V_{21}^{(k)} + v V_{12}^{(k)} - \omega V_{11}^{(k)} + \omega V_{22}^{(k)}\big) = \frac{u_x V_{21x}^{(k-1)}}{2\omega} - \frac{v_x V_{12x}^{(k-1)}}{2\omega}, \tag{2.6}$$

$$\partial_x\big(V_{11}^{(k)} + V_{22}^{(k)}\big) = 0. \tag{2.7}$$

From (2.4) and (2.5), we could easily prove that

$$\begin{aligned} V_{12}^{(k)}\big|_{(u,v)=(0,0)} &= \frac{-1}{2\omega}\partial_x V_{12}^{(k-1)}\big|_{(u,v)=(0,0)} \\ &= \cdots = \left(\frac{-1}{2\omega}\right)^k \partial_x^k V_{12}^{(0)}\big|_{(u,v)=(0,0)} = 0 \\ V_{21}^{(k)}\big|_{(u,v)=(0,0)} &= \frac{1}{2\omega}\partial_x V_{21}^{(k-1)}\big|_{(u,v)=(0,0)} \\ &= \cdots = \left(\frac{1}{2\omega}\right)^k \partial_x^k V_{21}^{(0)}\big|_{(u,v)=(0,0)} = 0. \end{aligned}$$

We use the condition $V_{11}^{(k)}\big|_{(u,v)=(0,0)} = V_{22}^{(k)}\big|_{(u,v)=(0,0)} = 0$ $(1 \leqslant k \leqslant n)$ to select the integration constant to be zero, then (2.6) and (2.7) are equivalent to

$$u V_{21}^{(k)} + v V_{12}^{(k)} - \omega V_{11}^{(k)} + \omega V_{22}^{(k)} = \int \frac{u_x V_{21x}^{(k-1)}}{2\omega} - \frac{v_x V_{12x}^{(k-1)}}{2\omega}\, \mathrm{d}x, \tag{2.8}$$

$$V_{11}^{(k)} + V_{22}^{(k)} = 0. \tag{2.9}$$

By using (2.4), (2.5), (2.8) and (2.9), we can get $V_{ij}^{(k)}$ from $V_{ij}^{(k-1)}$. A direct calculation gives

$$V_{11}^{(1)} = \tfrac{1}{2}(\omega_x\omega + uv_x), \qquad V_{12}^{(1)} = \tfrac{1}{2}(u\omega_x - u_x\omega),$$
$$V_{21}^{(1)} = \tfrac{1}{2}(v_x\omega - v\omega_x), \qquad V_{22}^{(1)} = \tfrac{1}{2}(\omega_x\omega + u_xv),$$
$$V_{11}^{(2)} = -\tfrac{1}{4}\left(\omega_{xx} + \tfrac{3}{2}\omega\left(u_xv_x + \omega_x^2\right)\right), \qquad V_{12}^{(2)} = \tfrac{1}{4}\left(u_{xx} + \tfrac{3}{2}u\left(u_xv_x + \omega_x^2\right)\right),$$
$$V_{21}^{(2)} = \tfrac{1}{4}\left(v_{xx} + \tfrac{3}{2}v\left(u_xv_x + \omega_x^2\right)\right), \qquad V_{22}^{(2)} = \tfrac{1}{4}\left(\omega_{xx} + \tfrac{3}{2}\omega\left(u_xv_x + \omega_x^2\right)\right).$$
$$\cdots \quad \cdots$$

So the soliton hierarchy associated with Heisenberg spectral problem (1.1) can be written as follows:

$$u_{t_n} = V_{12x}^{(n)} \qquad v_{t_n} = V_{21x}^{(n)}, \qquad n = 0, 1, 2, \ldots. \tag{2.10}$$

The first and second typical nonlinear systems ($n = 1, 2$) in the hierarchy are, respectively,

$$u_{t_1} = \tfrac{1}{2}(u\omega_x - u_x\omega)_x, \qquad v_{t_1} = \tfrac{1}{2}(v_x\omega - v\omega_x)_x, \tag{2.11}$$

and

$$u_{t_2} = \tfrac{1}{4}\left(u_{xx} + \tfrac{3}{2}u\left(u_xv_x + \omega_x^2\right)\right)_x, \qquad v_{t_2} = \tfrac{1}{4}\left(v_{xx} + \tfrac{3}{2}v\left(u_xv_x + \omega_x^2\right)\right)_x. \tag{2.12}$$

3. Darboux transformation

In this section, we will construct a Darboux transformation for the soliton hierarchy (2.10). The Darboux transformation is actually a special gauge transformation

$$\widetilde{\psi} = T\psi \tag{3.1}$$

of the solutions of the Lax pairs (1.1) and (2.1). It is required that $\widetilde{\psi}$ also satisfies Lax pairs (1.1) and (2.1) with some $\widetilde{U}$ and $\widetilde{V}^{(n)}$, i.e.

$$\widetilde{\psi}_x = \widetilde{U}\widetilde{\psi}, \qquad \widetilde{U} = (T_x + TU)T^{-1}, \tag{3.2}$$

$$\widetilde{\psi}_t = \widetilde{V}^{(n)}\widetilde{\psi}, \qquad \widetilde{V}^{(n)} = (T_t + TV^{(n)})T^{-1}. \tag{3.3}$$

By cross differentiating (3.2) and (3.3), we get

$$\widetilde{U}_t - \widetilde{V}^{(n)} + [\widetilde{U}, \widetilde{V}^{(n)}] = T\left(U_t - V_x^{(n)} + [U, V^{(n)}]\right)T^{-1}, \tag{3.4}$$

which implies that in order to make systems (2.10) invariant under the gauge transformation (3.1), we should require $\widetilde{U}$, $\widetilde{V}^{(n)}$ have the same forms as U, $V^{(n)}$, respectively. At the same time the old potentials u and v in U, $V^{(n)}$ will be mapped into new potentials $\widetilde{u}$ and $\widetilde{v}$ in $\widetilde{U}$, $\widetilde{V}^{(n)}$. This process can be done continually and usually it may yield a series of multisoliton solutions. Following the idea of [7], we can construct Darboux transformation for soliton hierarchy (2.10) as follows.

Let h_i be the solution of spectral problems (1.1) and (2.1) when $\lambda = \lambda_i$ ($i = 1, 2$; $\lambda_i \neq 0$). We construct a new matrix

$$H = (h_1, h_2).$$

From (1.1) and (2.1), we can get

$$H_x = U_1 H\Lambda, \qquad H_t = \sum_{k=0}^{n} V_k H\Lambda^{n-k+1}, \qquad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2). \tag{3.5}$$

Then we construct

$$T = Q_1\lambda + Q_0, \tag{3.6}$$

where

$$Q_0 = -I, \qquad Q_1 = H\Lambda^{-1}H^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{3.7}$$

It is easy to see that

$$Q_{1x} = H_x\Lambda^{-1}H^{-1} - H\Lambda^{-1}H^{-1}H_xH^{-1} = U_1 - Q_1U_1Q_1^{-1}, \tag{3.8}$$

and

$$\det Q_1 = AD - BC = \frac{1}{\lambda_1\lambda_2} \neq 0. \tag{3.9}$$

Substituting (3.6) into (3.2) and (3.3), by using (3.5), we can obtain

$$\widetilde{U} = \widetilde{U}_1\lambda, \qquad \widetilde{V}^{(n)} = \sum_{k=0}^{n} \widetilde{V}_k\lambda^{n-k+1}, \tag{3.10}$$

where

$$\begin{aligned} &\widetilde{U}_1 = Q_1U_1Q_1^{-1}, \qquad \widetilde{V}_0 = Q_1V_0Q_1^{-1}, \\ &\widetilde{V}_k = Q_1V_kQ_1^{-1} - V_{k-1}Q_1^{-1} + \widetilde{V}_{k-1}Q_1^{-1}, \qquad 1 \leqslant k \leqslant n. \end{aligned} \tag{3.11}$$

Next we will prove that $\widetilde{U}$ and $\widetilde{V}^{(n)}$ also have the same forms as U and $V^{(n)}$ after some transformations.

Proposition 1. *The matrix $\widetilde{U}$ determined by (3.10) has the same form as U, that is*

$$\widetilde{U} = \widetilde{U}_1\lambda = \begin{pmatrix} -\lambda\widetilde{\omega} & \lambda\widetilde{u} \\ \lambda\widetilde{v} & \lambda\widetilde{\omega} \end{pmatrix}, \qquad \widetilde{\omega}^2 + \widetilde{u}\widetilde{v} = 1, \tag{3.12}$$

where the transformations between u, v, ω and $\widetilde{u}, \widetilde{v}, \widetilde{\omega}$ are given by

$$\begin{aligned} \widetilde{u} &= \frac{2\omega AB + uA^2 - vB^2}{AD - BC}, \\ \widetilde{v} &= \frac{-2\omega CD - uC^2 + vD^2}{AD - BC}, \\ \widetilde{\omega} &= \frac{\omega(AD + BC) + uAC - vBD}{AD - BC}. \end{aligned} \tag{3.13}$$

A, B, C, D are determined by (3.7). The transformation $(\psi, u, v, \omega) \to (\widetilde{\psi}, \widetilde{u}, \widetilde{v}, \widetilde{\omega})$ is called a Darboux transformation of the spectral problem (1.1).

Proof. From (3.10) we can obtain

$$\begin{aligned} \widetilde{U}_1 &= Q_1U_1Q_1^{-1} \\ &= \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -\omega & u \\ v & \omega \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \\ &= \frac{1}{AD - BC} \begin{pmatrix} -\omega(AD + BC) - uAC + vBD & 2\omega AB + uA^2 - vB^2 \\ -2\omega CD - uC^2 + vD^2 & \omega(AD + BC) + uAC - vBD \end{pmatrix} \\ &= \begin{pmatrix} -\widetilde{\omega} & \widetilde{u} \\ \widetilde{v} & \widetilde{\omega} \end{pmatrix}. \end{aligned}$$

We can also find

$$\widetilde{\omega}^2 + \widetilde{u}\,\widetilde{v} = -\det \widetilde{U}_1 = -\det Q_1 \cdot \det U_1 \cdot \det Q_1^{-1} = 1.$$

So we get that after the transformation (3.13), $\widetilde{U}$ has the same form as U. The proof is completed. □

Next we shall prove that $\widetilde{V}^{(n)}$ also has the same form as $V^{(n)}$ under the transformations (3.1) and (3.13).

Proposition 2. *The matrix $\widetilde{V}^{(n)}$ determined by (3.10) has the same form as $V^{(n)}$ under the transformations (3.1) and (3.13).*

Proof. Because $V^{(n)} = \sum_{k=0}^{n} V_k \lambda^{n-k+1}$ and $\widetilde{V}^{(n)} = \sum_{k=0}^{n} \widetilde{V}_k \lambda^{n-k+1}$, we only need to prove that $\widetilde{V}_k$ has the same form as V_k after the transformations (3.1) and (3.13) $(0 \leqslant k \leqslant n)$.

First, it is easy to see that

$$V_0 = U_1, \qquad \widetilde{V}_0 = Q_1 V_0 Q_1^{-1} = \widetilde{U}_1.$$

From proposition 1, we can get $\widetilde{V}_0$ which has the same form as V_0.

Again by using (2.2) and (3.4), we can get

$$\widetilde{U}_t - \widetilde{V}_x^{(n)} + [\widetilde{U}, \widetilde{V}^{(n)}] = T\big(U_t - V_x^{(n)} + [U, V^{(n)}]\big)T^{-1} = 0. \tag{3.14}$$

We have proved that $\widetilde{U}$ has the same form as U, so $\widetilde{V}_k$ satisfies the same equations as V_k. Following a similar proof as in section 2, we could easily prove that $\widetilde{V}_{12}^{(k)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = \widetilde{V}_{21}^{(k)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = 0$. Next we only need to prove $\widetilde{V}_{11}^{(k)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = \widetilde{V}_{22}^{(k)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = 0$ $(1 \leqslant k \leqslant n)$.

In the following proof, we set $\widetilde{\omega}_0 = \widetilde{\omega}|_{(\widetilde{u},\widetilde{v})=(0,0)} = 1$ (the proof also holds for $\widetilde{\omega}_0 = -1$).

When $\widetilde{u} = 0, \widetilde{v} = 0$, we can get from (3.13) that

$$u = -\frac{2BD}{AD - BC}, \qquad v = \frac{2AC}{AD - BC}, \qquad \omega = \frac{AD + BC}{AD - BC}. \tag{3.15}$$

From (3.11), we have

$$\begin{aligned}\widetilde{V}_1|_{(\widetilde{u},\widetilde{v})=(0,0)} &= \begin{pmatrix} \widetilde{V}_{11}^{(1)} & \widetilde{V}_{12}^{(1)} \\ \widetilde{V}_{21}^{(1)} & \widetilde{V}_{22}^{(1)} \end{pmatrix}_{(\widetilde{u},\widetilde{v})=(0,0)} \\ &= Q_1 V_1 Q_1^{-1} - V_0 Q_1^{-1} + \widetilde{V}_0 Q_1^{-1}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} \\ &= Q_1 V_1 Q_1^{-1} - Q_1^{-1} \widetilde{V}_0 + \widetilde{V}_0 Q_1^{-1}\big|_{(\widetilde{u},\widetilde{v})=(0,0)},\end{aligned}$$

so we get

$$\widetilde{V}_{11}^{(1)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = \frac{1}{AD - BC}\big[(AD + BC)V_{11}^{(1)} + BDV_{21}^{(1)} - ACV_{12}^{(1)}\big], \tag{3.16}$$

$$\widetilde{V}_{22}^{(1)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = -\frac{1}{AD - BC}\big[(AD + BC)V_{11}^{(1)} + BDV_{21}^{(1)} - ACV_{12}^{(1)}\big]. \tag{3.17}$$

Substituting (3.15) into (3.16), (3.17) and choosing $k = 1$ in (2.8) and (2.9), we can get

$$\widetilde{V}_{11}^{(1)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = \widetilde{V}_{22}^{(1)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = 0. \tag{3.18}$$

If we suppose $\widetilde{V}_{ij}^{(k-1)}\big|_{(\widetilde{u},\widetilde{v})=(0,0)} = 0\ (1 \leqslant k \leqslant n)$ holds, again by using (3.11), we can get

$$\begin{aligned}\widetilde{V}_k|_{(\widetilde{u},\widetilde{v})=(0,0)} &= \begin{pmatrix} \widetilde{V}_{11}^{(k)} & \widetilde{V}_{12}^{(k)} \\ \widetilde{V}_{21}^{(k)} & \widetilde{V}_{22}^{(k)} \end{pmatrix}_{(\widetilde{u},\widetilde{v})=(0,0)} \\ &= Q_1 V_k Q_1^{-1} - V_{k-1} Q_1^{-1} + \widetilde{V}_{k-1} Q_1^{-1}\big|_{(\widetilde{u},\widetilde{v})=(0,0)}.\end{aligned}$$

From this equation and (3.15), we can obtain

$$\widetilde{V}_{11}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = \omega V_{11}^{(k)} - \frac{u}{2}V_{21}^{(k)} - \frac{v}{2}V_{12}^{(k)} - \frac{DV_{11}^{(k-1)} - CV_{12}^{(k-1)}}{AD-BC}, \tag{3.19}$$

$$\widetilde{V}_{22}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = -\omega V_{11}^{(k)} + \frac{u}{2}V_{21}^{(k)} + \frac{v}{2}V_{12}^{(k)} - \frac{-BV_{21}^{(k-1)} - AV_{11}^{(k-1)}}{AD-BC}, \tag{3.20}$$

$$\widetilde{V}_{12}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = \frac{-2ABV_{11}^{(k)} - B^2V_{21}^{(k)} + A^2V_{12}^{(k)}}{AD-BC} - \frac{-BV_{11}^{(k-1)} + AV_{12}^{(k-1)}}{AD-BC} = 0, \tag{3.21}$$

$$\widetilde{V}_{21}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = \frac{2CDV_{11}^{(k)} + D^2V_{21}^{(k)} - C^2V_{12}^{(k)}}{AD-BC} - \frac{DV_{21}^{(k-1)} + CV_{11}^{(k-1)}}{AD-BC} = 0. \tag{3.22}$$

By using (2.4), (2.5), (3.8), (3.15), (3.21) and (3.22), we can prove that when $\widetilde{u}=0, \widetilde{v}=0$

$$\partial_x\left(\frac{DV_{11}^{(k-1)} - CV_{12}^{(k-1)}}{AD-BC}\right) = -\frac{1}{2}\left[\frac{u_xV_{21x}^{(k-1)}}{2\omega} - \frac{v_xV_{12x}^{(k-1)}}{2\omega}\right], \tag{3.23}$$

$$\partial_x\left(\frac{-BV_{21}^{(k-1)} - AV_{11}^{(k-1)}}{AD-BC}\right) = \frac{1}{2}\left[\frac{u_xV_{21x}^{(k-1)}}{2\omega} - \frac{v_xV_{12x}^{(k-1)}}{2\omega}\right]. \tag{3.24}$$

Note the fact that $V_{ij}^{(s)}|_{(u,v)=(0,0)} = 0$ $(1 \leqslant s \leqslant n)$, so the integral constant must be zero, that is

$$\frac{DV_{11}^{(k-1)} - CV_{12}^{(k-1)}}{AD-BC} = -\int \frac{u_xV_{21x}^{(k-1)}}{4\omega} - \frac{v_xV_{12x}^{(k-1)}}{4\omega}\,\mathrm{d}x, \tag{3.25}$$

$$\frac{-BV_{21}^{(k-1)} - AV_{11}^{(k-1)}}{AD-BC} = \int \frac{u_xV_{21x}^{(k-1)}}{4\omega} - \frac{v_xV_{12x}^{(k-1)}}{4\omega}\,\mathrm{d}x. \tag{3.26}$$

Substituting (3.25), (3.26) into (3.19), (3.20) and using (2.8), we can finally get

$$\widetilde{V}_{11}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = \omega V_{11}^{(k)} - \frac{u}{2}V_{21}^{(k)} - \frac{v}{2}V_{12}^{(k)} + \int \frac{u_xV_{21x}^{(k-1)}}{4\omega} - \frac{v_xV_{12x}^{(k-1)}}{4\omega}\,\mathrm{d}x = 0, \tag{3.27}$$

$$\widetilde{V}_{22}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = -\omega V_{11}^{(k)} + \frac{u}{2}V_{21}^{(k)} + \frac{v}{2}V_{12}^{(k)} - \int \frac{u_xV_{21x}^{(k-1)}}{4\omega} - \frac{v_xV_{12x}^{(k-1)}}{4\omega}\,\mathrm{d}x = 0. \tag{3.28}$$

Thus, we proved that $\widetilde{V}_{ij}^{(k)}|_{(\widetilde{u},\widetilde{v})=(0,0)} = 0$ $(1 \leqslant k \leqslant n)$.

We proved that $\widetilde{V}^{(n)}$ and $V^{(n)}$ satisfy the same zero-curvature equation and the same boundary conditions. So they must have the same forms. The proof is completed. □

From propositions 1 and 2, we can get the following theorem:

Theorem 1. *The solutions* (u, v, ω) *of soliton hierarchy (2.10) are mapped into their new solutions* $(\widetilde{u}, \widetilde{v}, \widetilde{\omega})$ *under Darboux transformations (3.1) and (3.13), where* A, B, C, D *are given by (3.7).*

4. Applications of Darboux transformations

In this section, we will apply the Darboux transformation (3.13) to construct explicit solutions of the Heisenberg hierarchy (2.10). As usual we make the Darboux transformation starting from a special solution of (2.10). We start from $u = u_0$, $v = v_0$, $\omega = \omega_0$, and we choose

$$h_1^{(k)} = \begin{pmatrix} h_{11}^{(k)} \\ h_{12}^{(k)} \end{pmatrix}, \qquad h_2^{(k)} = \begin{pmatrix} h_{21}^{(k)} \\ h_{22}^{(k)} \end{pmatrix}, \qquad 1 \leqslant k \leqslant N, \tag{4.1}$$

as the solutions of Lax pairs (1.1) and (2.1) when $\lambda = \lambda_1^{(k)}$ and $\lambda = \lambda_2^{(k)}$. Then we could construct the multisoliton solutions of (2.10) as follows.

First, we construct

$$\begin{aligned} &H^{(1)} = \left(h_1^{(1)}, h_2^{(1)}\right), \qquad \Lambda^{(1)} = \begin{pmatrix} \lambda_1^{(1)} & 0 \\ 0 & \lambda_2^{(1)} \end{pmatrix}, \\ &Q_1^{(1)} = H^{(1)}(\Lambda^{(1)})^{-1}(H^{(1)})^{-1}. \end{aligned} \tag{4.2}$$

Then by the use of theorem 1, we can get the new solutions u_1, v_1, ω_1 of (2.10) from the following equation:

$$\begin{pmatrix} -\omega_1 & u_1 \\ v_1 & \omega_1 \end{pmatrix} = Q_1^{(1)} \begin{pmatrix} -\omega_0 & u_0 \\ v_0 & \omega_0 \end{pmatrix} \left(Q_1^{(1)}\right)^{-1}. \tag{4.3}$$

By the use of (3.1), (3.6) and after some calculations, we can get the solutions of Lax pairs (1.1) and (2.1), where $u = u_1$, $v = v_1$, $\omega = \omega_1$ and $\lambda = \lambda_1^{(2)}, \lambda_2^{(2)}$. These solutions can be expressed as follows:

$$\overline{h}_1^{(2)} = \Delta_1^{(2)} \begin{pmatrix} \begin{vmatrix} \frac{h_{11}^{(2)}}{\lambda_1^{(2)}} & \left[h_1^{(2)}, h_2^{(1)}\right] \\ \frac{h_{11}^{(1)}}{\lambda_1^{(1)}} & \left[h_1^{(1)}, h_2^{(1)}\right] \end{vmatrix} \\ \begin{vmatrix} \frac{h_{12}^{(2)}}{\lambda_1^{(2)}} & \left[h_1^{(2)}, h_2^{(1)}\right] \\ \frac{h_{12}^{(1)}}{\lambda_1^{(1)}} & \left[h_1^{(1)}, h_2^{(1)}\right] \end{vmatrix} \end{pmatrix}, \qquad \overline{h}_2^{(2)} = \Delta_2^{(2)} \begin{pmatrix} \begin{vmatrix} \frac{h_{21}^{(2)}}{\lambda_2^{(2)}} & \left[h_2^{(2)}, h_1^{(1)}\right] \\ \frac{h_{21}^{(1)}}{\lambda_2^{(1)}} & \left[h_2^{(1)}, h_1^{(1)}\right] \end{vmatrix} \\ \begin{vmatrix} \frac{h_{22}^{(2)}}{\lambda_2^{(2)}} & \left[h_2^{(2)}, h_1^{(1)}\right] \\ \frac{h_{22}^{(1)}}{\lambda_2^{(1)}} & \left[h_2^{(1)}, h_1^{(1)}\right] \end{vmatrix} \end{pmatrix},$$

where

$$\Delta_1^{(2)} = \frac{\lambda_1^{(2)}\left(\lambda_1^{(2)} - \lambda_2^{(1)}\right)}{\lambda_2^{(1)}\left[h_1^{(1)}, h_2^{(1)}\right]}, \qquad \Delta_2^{(2)} = \frac{\lambda_2^{(2)}\left(\lambda_2^{(2)} - \lambda_1^{(1)}\right)}{\lambda_1^{(1)}\left[h_2^{(1)}, h_1^{(1)}\right]},$$

$$\left[h_i^{(s)}, h_j^{(t)}\right] = \frac{h_{i1}^{(s)} h_{j2}^{(t)} - h_{i2}^{(s)} h_{j1}^{(t)}}{\lambda_i^{(s)} - \lambda_j^{(t)}}, \qquad i, j = 1, 2.$$

We construct

$$\begin{aligned} &H^{(2)} = \left(\overline{h}_1^{(2)}, \overline{h}_2^{(2)}\right), \qquad \Lambda^{(2)} = \begin{pmatrix} \lambda_1^{(2)} & 0 \\ 0 & \lambda_2^{(2)} \end{pmatrix}, \\ &Q_1^{(2)} = H^{(2)}(\Lambda^{(2)})^{-1}(H^{(2)})^{-1}. \end{aligned} \tag{4.4}$$

Then we can get the new solutions u_2, v_2, ω_2 of (2.10) from the following equation:

$$\begin{aligned} \begin{pmatrix} -\omega_2 & u_2 \\ v_2 & \omega_2 \end{pmatrix} &= Q_1^{(2)} \begin{pmatrix} -\omega_1 & u_1 \\ v_1 & \omega_1 \end{pmatrix} \left(Q_1^{(2)}\right)^{-1} \\ &= Q_1^{(2)} Q_1^{(1)} \begin{pmatrix} -\omega_0 & u_0 \\ v_0 & \omega_0 \end{pmatrix} \left(Q_1^{(1)}\right)^{-1} \left(Q_1^{(2)}\right)^{-1}. \end{aligned} \tag{4.5}$$

If we have done the Darboux transformation $N-1$ times, and got the solutions $u_{N-1}, v_{N-1}, \omega_{N-1}$ of system (2.10), we can express the solutions of Lax pairs (1.1) and (2.1) $\left(u=u_{N-1}, v=v_{N-1}, \omega=\omega_{N-1}, \lambda=\lambda_1^{(N)}, \lambda_2^{(N)}\right)$ as follows:

$$\overline{h}_1^{(N)} = \Delta_1^{(N)} \begin{pmatrix} \begin{vmatrix} \frac{1}{\lambda_1^{(N)}} h_{11}^{(N)} & \left[h_1^{(N)}, h_2^{(1)}\right] & \cdots & \left[h_1^{(N)}, h_2^{(N-1)}\right] \\ \frac{1}{\lambda_1^{(1)}} h_{11}^{(1)} & \left[h_1^{(1)}, h_2^{(1)}\right] & \cdots & \left[h_1^{(1)}, h_2^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_1^{(N-1)}} h_{11}^{(N-1)} & \left[h_1^{(N-1)}, h_2^{(1)}\right] & \cdots & \left[h_1^{(N-1)}, h_2^{(N-1)}\right] \end{vmatrix} \\ \begin{vmatrix} \frac{1}{\lambda_1^{(N)}} h_{12}^{(N)} & \left[h_1^{(N)}, h_2^{(1)}\right] & \cdots & \left[h_1^{(N)}, h_2^{(N-1)}\right] \\ \frac{1}{\lambda_1^{(1)}} h_{12}^{(1)} & \left[h_1^{(1)}, h_2^{(1)}\right] & \cdots & \left[h_1^{(1)}, h_2^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_1^{(N-1)}} h_{12}^{(N-1)} & \left[h_1^{(N-1)}, h_2^{(1)}\right] & \cdots & \left[h_1^{(N-1)}, h_2^{(N-1)}\right] \end{vmatrix} \end{pmatrix},$$

$$\overline{h}_2^{(N)} = \Delta_2^{(N)} \begin{pmatrix} \begin{vmatrix} \frac{1}{\lambda_2^{(N)}} h_{21}^{(N)} & \left[h_2^{(N)}, h_1^{(1)}\right] & \cdots & \left[h_2^{(N)}, h_1^{(N-1)}\right] \\ \frac{1}{\lambda_2^{(1)}} h_{21}^{(1)} & \left[h_2^{(1)}, h_1^{(1)}\right] & \cdots & \left[h_2^{(1)}, h_1^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_2^{(N-1)}} h_{21}^{(N-1)} & \left[h_2^{(N-1)}, h_1^{(1)}\right] & \cdots & \left[h_2^{(N-1)}, h_1^{(N-1)}\right] \end{vmatrix} \\ \begin{vmatrix} \frac{1}{\lambda_2^{(N)}} h_{22}^{(N)} & \left[h_2^{(N)}, h_1^{(1)}\right] & \cdots & \left[h_2^{(N)}, h_1^{(N-1)}\right] \\ \frac{1}{\lambda_2^{(1)}} h_{22}^{(1)} & \left[h_2^{(1)}, h_1^{(1)}\right] & \cdots & \left[h_2^{(1)}, h_1^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\lambda_2^{(N-1)}} h_{22}^{(N-1)} & \left[h_2^{(N-1)}, h_1^{(1)}\right] & \cdots & \left[h_2^{(N-1)}, h_1^{(N-1)}\right] \end{vmatrix} \end{pmatrix},$$

where

$$\Delta_1^{(N)} = \frac{\lambda_1^{(N)}\left(\frac{\lambda_1^{(N)}}{\lambda_2^{(1)}}-1\right)\cdots\left(\frac{\lambda_1^{(N)}}{\lambda_2^{(N-1)}}-1\right)}{\begin{vmatrix} \left[h_1^{(1)}, h_2^{(1)}\right] & \left[h_1^{(1)}, h_2^{(2)}\right] & \cdots & \left[h_1^{(1)}, h_2^{(N-1)}\right] \\ \left[h_1^{(2)}, h_2^{(1)}\right] & \left[h_1^{(2)}, h_2^{(2)}\right] & \cdots & \left[h_1^{(2)}, h_2^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[h_1^{(N-1)}, h_2^{(1)}\right] & \left[h_1^{(N-1)}, h_2^{(2)}\right] & \cdots & \left[h_1^{(N-1)}, h_2^{(N-1)}\right] \end{vmatrix}},$$

$$\Delta_2^{(N)} = \frac{\lambda_2^{(N)}\left(\frac{\lambda_2^{(N)}}{\lambda_1^{(1)}}-1\right)\cdots\left(\frac{\lambda_2^{(N)}}{\lambda_1^{(N-1)}}-1\right)}{\begin{vmatrix} \left[h_2^{(1)}, h_1^{(1)}\right] & \left[h_2^{(1)}, h_1^{(2)}\right] & \cdots & \left[h_2^{(1)}, h_1^{(N-1)}\right] \\ \left[h_2^{(2)}, h_1^{(1)}\right] & \left[h_2^{(2)}, h_1^{(2)}\right] & \cdots & \left[h_2^{(2)}, h_1^{(N-1)}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[h_2^{(N-1)}, h_1^{(1)}\right] & \left[h_2^{(N-1)}, h_1^{(2)}\right] & \cdots & \left[h_2^{(N-1)}, h_1^{(N-1)}\right] \end{vmatrix}}.$$

We construct

$$H^{(N)} = \left(\overline{h}_1^{(N)}, \overline{h}_2^{(N)}\right), \qquad \Lambda^{(N)} = \begin{pmatrix} \lambda_1^{(N)} & 0 \\ 0 & \lambda_2^{(N)} \end{pmatrix},$$
$$Q_1^{(N)} = H^{(N)}(\Lambda^{(N)})^{-1}(H^{(N)})^{-1}. \tag{4.6}$$

Then we can get the new solutions u_N, v_N, ω_N for (2.10) from the following equation:

$$\begin{pmatrix} -\omega_N & u_N \\ v_N & \omega_N \end{pmatrix} = Q_1^{(N)} \begin{pmatrix} -\omega_{N-1} & u_{N-1} \\ v_{N-1} & \omega_{N-1} \end{pmatrix} \left(Q_1^{(N)}\right)^{-1}$$
$$= Q_1^{(N)} \cdots Q_1^{(1)} \begin{pmatrix} -\omega_0 & u_0 \\ v_0 & \omega_0 \end{pmatrix} \left(Q_1^{(1)}\right)^{-1} \cdots \left(Q_1^{(N)}\right)^{-1}. \tag{4.7}$$

This process can be done continually and yields a series of soliton solutions of the Heisenberg hierarchy in theory.

In the end, we will give a simple example. We will construct the 1-soliton solutions for Heisenberg hierarchy (2.10). Substituting $u = 0, v = 0, \omega = 1$ into the Lax pairs (1.1) and (2.1), we choose two basic solutions corresponding to $\lambda = \lambda_1$ and $\lambda = \lambda_2$ as follows:

$$h_1 = \begin{pmatrix} \mathrm{e}^{-\xi_1} \\ \mathrm{e}^{\xi_1} \end{pmatrix}, \qquad h_2 = \begin{pmatrix} -\mathrm{e}^{-\xi_2} \\ \mathrm{e}^{\xi_2} \end{pmatrix}, \tag{4.8}$$

where $\xi_i = \lambda_i\left(x + \lambda_i^n t\right), i = 1, 2$. Then, we can construct

$$H = (h_1, h_2) = \begin{pmatrix} \mathrm{e}^{-\xi_1} & -\mathrm{e}^{-\xi_2} \\ \mathrm{e}^{\xi_1} & \mathrm{e}^{\xi_2} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = H\Lambda^{-1}H^{-1}$$
$$= \begin{pmatrix} \lambda_2\,\mathrm{e}^{\xi_2-\xi_1} + \lambda_1\,\mathrm{e}^{\xi_1-\xi_2} & (\lambda_2-\lambda_1)\,\mathrm{e}^{-(\xi_1+\xi_2)} \\ (\lambda_2-\lambda_1)\,\mathrm{e}^{\xi_1+\xi_2} & \lambda_1\,\mathrm{e}^{\xi_2-\xi_1} + \lambda_2\,\mathrm{e}^{\xi_1-\xi_2} \end{pmatrix} \cdot \frac{1}{\lambda_1\lambda_2(\mathrm{e}^{\xi_2-\xi_1} + \mathrm{e}^{\xi_1-\xi_2})}.$$

Thus, from (3.13) we can get

$$\widetilde{u} = \frac{2AB}{AD-BC} = \frac{2(\lambda_2-\lambda_1)(\lambda_2\,\mathrm{e}^{-2\xi_1} + \lambda_1\,\mathrm{e}^{-2\xi_2})}{\lambda_1\lambda_2(\mathrm{e}^{\xi_2-\xi_1} + \mathrm{e}^{\xi_1-\xi_2})^2},$$
$$\widetilde{v} = \frac{-2CD}{AD-BC} = \frac{2(\lambda_2-\lambda_1)(\lambda_1\,\mathrm{e}^{2\xi_2} + \lambda_2\,\mathrm{e}^{2\xi_1})}{\lambda_1\lambda_2(\mathrm{e}^{\xi_2-\xi_1} + \mathrm{e}^{\xi_1-\xi_2})^2}, \tag{4.9}$$
$$\widetilde{\omega} = \frac{AD+BC}{AD-BC} = 1 + \frac{2(\lambda_1-\lambda_2)^2}{\lambda_1\lambda_2(\mathrm{e}^{\xi_2-\xi_1} + \mathrm{e}^{\xi_1-\xi_2})^2}.$$

If we choose $n = 1$ and $n = 2$ in (4.9), we can get the 1-soliton solution for systems (2.11) and (2.12).

Acknowledgments

The author expresses his gratitude to Professor E G Fan for his enthusiastic guidance. He also thanks the referees for their helpful suggestions. The work described in this paper was partially supported by a National Key Basic Research Project of China (2004CB318000), the National Science Foundation of China (10371023) and Shanghai Shuguang Project of China (02SG02).

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